

SOBOLEV INEQUALITIES AND THE $\bar{\partial}$ -NEUMANN OPERATOR

FRIEDRICH HASLINGER

ABSTRACT. We study a complex valued version of the Sobolev inequalities and its relationship to compactness of the $\bar{\partial}$ -Neumann operator. For this purpose we use an abstract characterization of compactness derived from a general description of precompact subsets in L^2 -spaces. Finally we remark that the $\bar{\partial}$ -Neumann operator can be continuously extended provided a subelliptic estimate holds.

1. INTRODUCTION.

Let Ω be a bounded open set in \mathbb{R}^n , and k a nonnegative integer. We denote by $W^k(\Omega)$ the Sobolev space

$$W^k(\Omega) = \{f \in L^2(\Omega) : \partial^\alpha f \in L^2(\Omega), |\alpha| \leq k\},$$

where the derivatives are taken in the sense of distributions and endow the space with the norm

$$\|f\|_{k,\Omega} = \left(\sum_{|\alpha| \leq k} \int_{\Omega} |\partial^\alpha f|^2 d\lambda \right)^{1/2},$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multiindex, $|\alpha| = \sum_{j=1}^n \alpha_j$ and

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}.$$

$W^k(\Omega)$ is a Hilbert space. If $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain with a \mathcal{C}^1 boundary, the Rellich-Kondrachov lemma says that for $n > 2$ one has

$$W^1(\Omega) \subset L^r(\Omega), \quad r \in [1, 2n/(n-2))$$

and that the imbedding is also compact; for $n = 2$ one can take $r \in [1, \infty)$ (see for instance [4]), in particular, there exists a constant C_r such that

$$(1.1) \quad \|f\|_r \leq C_r \|f\|_{1,\Omega},$$

for each $f \in W^1(\Omega)$, where

$$\|f\|_r = \left(\int_{\Omega} |f|^r d\lambda \right)^{1/r}.$$

Now let $\Omega \subseteq \mathbb{C}^n (\cong \mathbb{R}^{2n})$ be a smoothly bounded pseudoconvex domain. We consider the $\bar{\partial}$ -complex

$$(1.2) \quad L^2(\Omega) \xrightarrow{\bar{\partial}} L^2_{(0,1)}(\Omega) \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} L^2_{(0,n)}(\Omega) \xrightarrow{\bar{\partial}} 0,$$

2010 *Mathematics Subject Classification.* Primary 32W05; Secondary 30H20, 35P10.

Key words and phrases. $\bar{\partial}$ -Neumann problem, Sobolev inequalities, compactness.

Partially supported by the FWF-grant P23664.

where $L^2_{(0,q)}(\Omega)$ denotes the space of $(0, q)$ -forms on Ω with coefficients in $L^2(\Omega)$. The $\bar{\partial}$ -operator on $(0, q)$ -forms is given by

$$(1.3) \quad \bar{\partial} \left(\sum_J ' a_J d\bar{z}_J \right) = \sum_{j=1}^n \sum_J ' \frac{\partial a_J}{\partial \bar{z}_j} d\bar{z}_j \wedge d\bar{z}_J,$$

where \sum' means that the sum is only taken over strictly increasing multi-indices J . The derivatives are taken in the sense of distributions, and the domain of $\bar{\partial}$ consists of those $(0, q)$ -forms for which the right hand side belongs to $L^2_{(0,q+1)}(\Omega)$. So $\bar{\partial}$ is a densely defined closed operator, and therefore has an adjoint operator from $L^2_{(0,q+1)}(\Omega)$ into $L^2_{(0,q)}(\Omega)$ denoted by $\bar{\partial}^*$.

We consider the $\bar{\partial}$ -complex

$$(1.4) \quad L^2_{(0,q-1)}(\Omega) \xrightarrow[\bar{\partial}^*]{\bar{\partial}} L^2_{(0,q)}(\Omega) \xrightarrow[\bar{\partial}^*]{\bar{\partial}} L^2_{(0,q+1)}(\Omega),$$

for $1 \leq q \leq n-1$.

We remark that a $(0, q+1)$ -form $u = \sum_J' u_J d\bar{z}_J$ belongs to $\mathcal{C}^\infty_{(0,q+1)}(\bar{\Omega}) \cap \text{dom}(\bar{\partial}^*)$ if and only if

$$(1.5) \quad \sum_{k=1}^n u_{kK} \frac{\partial r}{\partial z_k} = 0$$

on $b\Omega$ for all K with $|K| = q$, where r is a defining function of Ω with $|\nabla r(z)| = 1$ on the boundary $b\Omega$. (see for instance [14])

The complex Laplacian $\square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$, defined on the domain

$$\text{dom}(\square) = \{u \in L^2_{(0,q)}(\Omega) : u \in \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*), \bar{\partial}u \in \text{dom}(\bar{\partial}^*), \bar{\partial}^*u \in \text{dom}(\bar{\partial})\}$$

acts as an unbounded, densely defined, closed and self-adjoint operator on $L^2_{(0,q)}(\Omega)$, for $1 \leq q \leq n$, which means that $\square = \square^*$ and $\text{dom}(\square) = \text{dom}(\square^*)$.

Note that

$$(1.6) \quad (\square u, u) = (\bar{\partial}\bar{\partial}^*u + \bar{\partial}^*\bar{\partial}u, u) = \|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2,$$

for $u \in \text{dom}(\square)$.

If Ω is a smoothly bounded pseudoconvex domain in \mathbb{C}^n , the so-called basic estimate says that

$$(1.7) \quad \|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2 \geq c \|u\|^2,$$

for each $u \in \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$, $c > 0$.

This estimate implies that $\square : \text{dom}(\square) \rightarrow L^2_{(0,q)}(\Omega)$ is bijective and has a bounded inverse

$$N : L^2_{(0,q)}(\Omega) \rightarrow \text{dom}(\square).$$

N is called $\bar{\partial}$ -Neumann operator. In addition

$$(1.8) \quad \|Nu\| \leq \frac{1}{c} \|u\|.$$

A different approach to the $\bar{\partial}$ -Neumann operator is related to the quadratic form

$$Q(u, v) = (\bar{\partial}u, \bar{\partial}v) + (\bar{\partial}^*u, \bar{\partial}^*v).$$

For this purpose we consider the embedding

$$j : \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*) \longrightarrow L^2_{(0,q)}(\Omega),$$

where $\text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$ is endowed with the graph-norm

$$u \mapsto (\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2)^{1/2}.$$

The graph-norm stems from the inner product $Q(u, v)$. The basic estimates (1.7) imply that j is a bounded operator with operator norm

$$\|j\| \leq \frac{1}{\sqrt{c}}.$$

By (1.7) it follows in addition that $\text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$ endowed with the graph-norm $u \mapsto (\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2)^{1/2}$ is a Hilbert space.

The $\bar{\partial}$ -Neumann operator N can be written in the form

$$(1.9) \quad N = j \circ j^*,$$

details may be found in [14].

2. COMPACTNESS AND SOBOLEV INEQUALITIES.

Here we apply a general characterization of compactness of the $\bar{\partial}$ -Neumann operator N using a description of precompact subsets in L^2 -spaces (see [10]).

Theorem 2.1. *Let $\Omega \subset\subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain. The $\bar{\partial}$ -Neumann operator N is compact if and only if for each $\epsilon > 0$ there exists $\omega \subset\subset \Omega$ such that*

$$\int_{\Omega \setminus \omega} |u(z)|^2 d\lambda(z) \leq \epsilon (\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2)$$

for each $u \in \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$.

Now let

$$\mathcal{W}_{0,q}^1(\Omega) := \{u \in L^2_{(0,q)}(\Omega) : u \in \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)\}$$

endowed with graph norm. As already mentioned above, this "complex" version of a Sobolev space $\mathcal{W}_{0,q}^1(\Omega)$ is a Hilbert space.

It appears to be interesting to compare the standard Sobolev imbedding

$$W^1(\Omega) \subset L^r(\Omega), \quad r \in [1, 2n/(n-1))$$

where the derivatives are taken with respect of the real variables $x_j = \Re z_j$ and $y_j = \Im z_j$ for $j = 1, \dots, n$, with the imbedding of the space $\mathcal{W}_{0,q}^1(\Omega)$ endowed with graph norm, into $L^r_{(0,q)}(\Omega)$. We have the following result

Theorem 2.2. *If $\Omega \subset\subset \mathbb{C}^n$ is a smoothly bounded pseudoconvex domain and the inequality*

$$(2.1) \quad \|u\|_r \leq C((\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2)^{1/2})$$

for some $r > 2$ and for all $u \in \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$ holds, then the $\bar{\partial}$ -Neumann operator

$$N : L^2_{(0,q)}(\Omega) \longrightarrow L^2_{(0,q)}(\Omega)$$

is compact.

Proof. To show this we have to check that the unit ball in $\mathcal{W}_{(0,q)}^1(\Omega)$ is precompact in $L_{(0,q)}^2(\Omega)$. By Proposition 2.1, we have to show that for each $\epsilon > 0$ there exists $\omega \subset\subset \Omega$ such that

$$\int_{\Omega \setminus \omega} |u(z)|^2 d\lambda(z) < \epsilon^2,$$

for all u in the unit ball of $\mathcal{W}_{(0,q)}^1(\Omega)$.

By (2.1) and Hölder's inequality we have

$$\begin{aligned} \left(\int_{\Omega \setminus \omega} |u(z)|^2 d\lambda(z) \right)^{\frac{1}{2}} &\leq \left(\int_{\Omega \setminus \omega} |u(z)|^r d\lambda(z) \right)^{\frac{1}{r}} \cdot |\Omega \setminus \omega|^{\frac{1}{2} - \frac{1}{r}} \\ &\leq C |\Omega \setminus \omega|^{\frac{1}{2} - \frac{1}{r}}. \end{aligned}$$

Now we can choose $\omega \subset\subset \Omega$ such that the last term is $< \epsilon$. \square

In the following Theorem we suppose that a so-called subelliptic estimate holds. Subelliptic estimates are related to the geometric notion of finite type. We remark that the $\bar{\partial}$ -Neumann problem for smoothly bounded strictly pseudoconvex domains is subelliptic with a gain of one derivative for N which is considerably stronger than compactness.

Theorem 2.3. *Let Ω be a bounded pseudoconvex domain in \mathbb{C}^n with boundary of class \mathcal{C}^∞ . Suppose that $0 < \epsilon \leq 1/2$ and that*

$$\text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*) \subseteq W_{(0,q)}^\epsilon(\Omega),$$

and that there exists a constant $C > 0$ such that

$$(2.2) \quad \|u\|_{\epsilon, \Omega} \leq C(\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2)^{1/2},$$

for all $u \in \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^)$, where $W_{(0,q)}^\epsilon(\Omega)$ is the standard ϵ -Sobolev space. Then the $\bar{\partial}$ -Neumann operator*

$$N : L_{(0,q)}^2(\Omega) \longrightarrow L_{(0,q)}^2(\Omega)$$

is compact and N can be continuously extended as an operator

$$\tilde{N} : L_{(0,q)}^{\frac{2n}{n+\epsilon}}(\Omega) \longrightarrow L_{(0,q)}^{\frac{2n}{n-\epsilon}}(\Omega),$$

which means that there is a constant $C > 0$ such that

$$(2.3) \quad \|\tilde{N}u\|_{\frac{2n}{n-\epsilon}} \leq C \|u\|_{\frac{2n}{n+\epsilon}},$$

for each $u \in L_{(0,q)}^{\frac{2n}{n+\epsilon}}(\Omega)$.

Proof. We use the continuous imbedding for the space $W^\epsilon(\Omega)$:

$$W^\epsilon(\Omega) \longrightarrow L^r(\Omega),$$

for $2 \leq r \leq 2n/(n - \epsilon)$, (see [1], Theorem 7.57). Hence we can choose $r_0 > 2$ to get

$$\text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*) \subseteq W_{(0,q)}^\epsilon(\Omega) \subseteq L_{(0,q)}^{r_0}(\Omega),$$

and we can apply Theorem 2.2.

To show that N extends continuously recall that $N = j \circ j^*$, where

$$j : \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*) \longrightarrow L_{(0,q)}^2(\Omega),$$

see [14]. In our case j is a continuous operator into $L^{\frac{2n}{n-\epsilon}}_{(0,q)}(\Omega)$, hence

$$j^* : L^{\frac{2n}{n+\epsilon}}_{(0,q)}(\Omega) \longrightarrow \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*),$$

which proves the assertion. \square

St. Krantz [12], R. Beals, P.C. Greiner and N.K. Stanton [2], I.Lieb and R.M. Range [13], and A. Bonami and N. Sibony [3] proved L^p -estimates and Lipschitz estimates for solution operators of the inhomogeneous $\bar{\partial}$ -equation and the $\bar{\partial}$ -Neumann operator using integral representations for the kernel of these operators, but without relationship to compactness and continuous extendability.

Remark 2.4. *If Ω is a bounded strictly pseudoconvex domain in \mathbb{C}^n with boundary of class C^∞ , then (2.2) is satisfied for $\epsilon = 1/2$ (see [14], Proposition 3.1). D'Angelo ([8], [9]) and Catlin [5], [6], [7]) give a characterization of when a subelliptic estimate holds in terms of the geometric notion of finite type, see also [14].*

Corollary 2.5. *Let Ω be a smooth bounded pseudoconvex domain in \mathbb{C}^n , $n \geq 2$. Let $P \in b\Omega$ and assume that there is an m -dimensional complex manifold $M \subset b\Omega$ through P ($m \geq 1$), and $b\Omega$ is strictly pseudoconvex at P in the directions transverse to M (this condition is void when $n = 2$). Then (2.1) is not satisfied for $(0, q)$ -forms with $1 \leq q \leq m$.*

Proof. Theorem 4.21 of [14] gives that the $\bar{\partial}$ -Neumann operator fails to be compact on $(0, q)$ -forms with $1 \leq q \leq m$. Hence we can again apply Proposition 2.2 to get the desired result. \square

Remark 2.6. *If the Levi form of the defining function of Ω is known to have at most one degenerate eigenvalue at each point (the eigenvalue zero has multiplicity at most 1), a disk in the boundary is an obstruction to compactness of N for $(0, 1)$ -forms. A special case of this is implicit in [11] for domains fibered over a Reinhardt domain in \mathbb{C}^2 .*

ACKNOWLEDGEMENT

The author wishes to express his gratitude to the referee for helpful suggestions.

REFERENCES

1. R.A. Adams and J.J.F. Fournier, *Sobolev spaces*, Pure and Applied Math., vol. 140, Academic Press, 2006.
2. R. Beals, P.C. Greiner, and N.K. Stanton, *L^p and Lipschitz estimates for the $\bar{\partial}$ -equation the $\bar{\partial}$ -Neumann problem*, Math. Ann. **277** (1987), 185–196.
3. A. Bonami and N. Sibony, *Sobolev embedding in \mathbb{C}^n and the $\bar{\partial}$ -equation*, J. Geom. Anal. **1** (1991), 307–327.
4. H. Brezis, *Analyse fonctionnelle, théorie et applications*, Masson, Paris, 1983.
5. D.W. Catlin, *Necessary conditions for subellipticity of the $\bar{\partial}$ -Neumann problem*, Ann. of Math. **117** (1983), 147–171.
6. ———, *Boundary invariants of pseudoconvex domains*, Ann. of Math. **120** (1984), 529–586.
7. ———, *Subelliptic estimates for the $\bar{\partial}$ -Neumann problem on pseudoconvex domains*, Ann. of Math. **126** (1987), 131–191.
8. J.P. D'Angelo, *Finite type conditions for real hypersurfaces*, J. Differential Geom. **14** (1979), 59–66.
9. ———, *Real hypersurfaces, orders of contact, and applications*, Ann. of Math. **115** (1979), 615–637.
10. F. Haslinger, *Compactness for the $\bar{\partial}$ -Neumann problem- a functional analysis approach*, Collectanea Math. **62** (2011), 121–129.
11. M. Kim, *Inheritance of noncompactness of the $\bar{\partial}$ -Neumann problem*, J. Math. Anal. Appl. **302** (2005), 450–456.

12. St. Krantz, *Optimal Lipschitz and L^p estimates for the equation $\bar{\partial}u = f$ on strongly pseudoconvex domains*, Math. Ann. **219** (1976), 233–260.
13. I. Lieb and R.M. Range, *Integral representations and estimates in the theory of the $\bar{\partial}$ -Neumann problem*, Ann. of Math. **123** (1986), 265–301.
14. E. Straube, *The L^2 -Sobolev theory of the $\bar{\partial}$ -Neumann problem*, ESI Lectures in Mathematics and Physics, EMS, 2010.

F. HASLINGER: FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT WIEN, OSKAR-MORGENSTERN-PLATZ 1, A-1090 WIEN, AUSTRIA

E-mail address: `friedrich.haslinger@univie.ac.at`